4.3 – Spanning Sets

Definition: (generalizes linear combination from Section 3.1)
If $\mathbf{w}$ is a vector in a vector space V, then $\mathbf{w}$ is said to be a **linear combination** of the vectors $\mathbf{v}_1,\ \mathbf{v}_2,\ \ldots,\ \mathbf{v}_r$ in V if $\mathbf{w}$ can be expressed in the form $\mathbf{w} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \ldots + k_r\mathbf{v}_r$, where each k_i is a scalar. The scalars are called the **coefficients** of the linear combination. In the case where $r = 1$, we have $\mathbf{w} = k_1\mathbf{v}_1$, in which case the linear combination is a scalar multiple of the vector.

#2 Express the following as linear combinations of $\mathbf{u} = (2,\ 1,\ 4)$, $\mathbf{v} = (1,\ -1,\ 3)$, and $\mathbf{w} = (3,\ 2,\ 5)$.
a. $(-9,\ -7,\ -15)$
b. $(6,\ 11,\ 6)$
c. $(0,\ 0,\ 0)$

<u>Definition:</u> The **span** of a nonempty set $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_r\}$ of vectors is the set of all possible linear combinations of vectors in S, denoted by $\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_r\}$ or $\operatorname{span}(S)$. If $W = \operatorname{span}(S)$, then we say that W is **spanned** by S.

#9 Determine whether the following polynomials span P_2.

$\mathbf{p}_1 = 1 - x + 2x^2$, $\mathbf{p}_2 = 3 + x$, $\mathbf{p}_3 = 5 - x + 4x^2$, $\mathbf{p}_4 = -2 - 2x + 2x^2$

#12 Let $T_A : R^2 \to R^2$ be multiplication by A. Determine whether the vector $\mathbf{u} = (1,\ 2)$ is in the span of $\{T_A(\mathbf{e}_1),\ T_A(\mathbf{e}_2)\}$.

a. $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$

b. $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

#16 Let W be the solution space to the system $A\mathbf{x} = \mathbf{0}$. Determine whether the set $\{\mathbf{u},\ \mathbf{v}\}$ spans W.

$$A = \begin{bmatrix} 0 & 1 & -1 & 1 \\ 0 & 2 & -2 & 2 \\ 0 & 3 & -3 & 3 \end{bmatrix}$$

a. $\mathbf{u} = (1,\ 1,\ 1,\ 0)$, $\mathbf{v} = (0,\ -1,\ 0,\ 1)$

b. $\mathbf{u} = (0,\ 1,\ 1,\ 0)$, $\mathbf{v} = (1,\ 0,\ 1,\ 1)$

Theorem 4.3.1 If $S = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_r\}$ is a nonempty set of vectors in a vector space V, then:

a) The set W of all possible linear combinations of the vectors in S is a subspace of V.

b) The set W in part (a) is the "smallest" subspace of V that contains all of the vectors in S in the sense that any other subspace that contains those vectors contains W.

Theorem 4.3.2 If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ and $S' = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_k\}$ are nonempty sets of vectors in a vector space V, then $\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\} = \operatorname{span}\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_k\}$ if and only if each vector in S is a linear combination of those in S', and each vector in S' is a linear combination of those in S.

#20 Let $\mathbf{v}_1 = (1,\ 6,\ 4)$, $\mathbf{v}_2 = (2,\ 4,\ -1)$, $\mathbf{v}_3 = (-1,\ 2,\ 5)$, and $\mathbf{w}_1 = (1,\ -2,\ -5)$, $\mathbf{w}_2 = (0,\ 8,\ 9)$. Use Theorem 4.3.2 to show that $\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \operatorname{span}\{\mathbf{w}_1, \mathbf{w}_2\}$.